Appendix C

The Vector or Cross Product

We saw in Appendix B that the dot product of two vectors is a scalar quantity that is a maximum when the two vectors are parallel and is zero if the two vectors are normal or perpendicular to each other. We now discuss another kind of vector multiplication called the *vector* or *cross product*, which is a vector quantity that is a maximum when the two vectors are normal to each other and is zero if they are parallel.

Prerequisite knowledge:
Appendix B – The Scalar or Dot Product

C.1 Definition of the Cross Product

The vector or cross product of two vectors is written as $\mathbf{A}\times\mathbf{B}$ and reads "A cross B." It is defined to be a third vector **C** such that $\mathbf{A}\times\mathbf{B}=\mathbf{C}$, where the magnitude of **C** is

$$C=\left|\mathbf{C}\right|=AB\sin\phi \qquad \text{(C.1)}$$

and the direction of **C** is perpendicular to both **A** and **B** in a right-handed sense as shown in Fig. C.1. ϕ is the smaller angle between **A** and **B** and the direction of **C** is found by the following rule. Extend the fingers of your right hand along **A** and then curl them toward **B** as if you were rotating **A** through ϕ. Your thumb will then point in the direction of **C**. The vector product $\mathbf{B}\times\mathbf{A}$ has a magnitude $BA\sin\phi$ but its direction, found by rotating **B** into **A** through ϕ, is opposite to that of **C**. Therefore,

$$\mathbf{B}\times\mathbf{A}=-\mathbf{C}=-\left(\mathbf{A}\times\mathbf{B}\right) \qquad \text{(C.2)}$$

and the commutative law does not hold for the cross product.

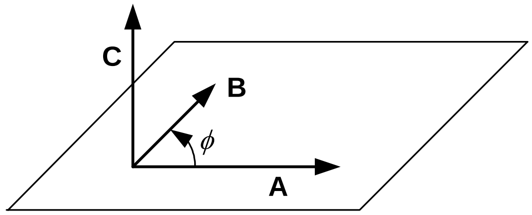

Figure C.1 $\mathbf{A}\times\mathbf{B}=\mathbf{C}$

C.2 Distributive Law for the Cross Product

The distributive law $\mathbf{A}\times(\mathbf{B}+\mathbf{C})=(\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})$ holds in general for the cross product and is illustrated for the special case shown in Fig. C.2 where **A**, B, and **C** all lie in the *x-y* plane and **D** = **B** + **C**. The +*z* direction is out of the paper so from the right-hand rule $\mathbf{A}\times\mathbf{D}$ is into the paper or in the $-\mathbf{k}$ direction. Similarly $\mathbf{A}\times\mathbf{C}$ is in the +**k** direction and $\mathbf{A}\times\mathbf{B}$ is in the $-\mathbf{k}$ direction. We can then write

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\times\mathbf{D}\\ &= -\mathbf{k}AD\sin(\theta-\phi)\\ &= -\mathbf{k}AD(\sin\theta\cos\phi-\cos\theta\sin\phi)\end{aligned}$$

But since $B = D\cos\phi$ and $C = D\sin\phi$

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= -\mathbf{k}AB\sin\theta+\mathbf{k}AC\cos\theta\\ &= (\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})\end{aligned}$$

since $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$ and the angle between **A** and **C** is $\left(\frac{\pi}{2}-\theta\right)$.

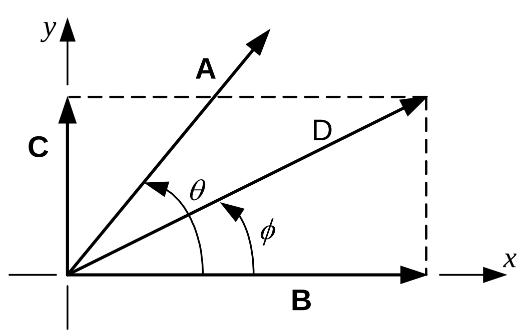

Figure C.2 $\mathbf{A}\times(\mathbf{B}+\mathbf{C})=(\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})$

C.3 Cross Product and Vector Components

We now wish to find the components of $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ in the rectangular coordinate system shown in Fig. C.3 if we know the components of **A** and **B**. From the definition of the cross product, if two vectors are parallel, then $\phi = 0$, $\sin\phi=0$, and their cross product is zero. In particular, the cross product of a vector with itself is always zero. Therefore $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$.

If two vectors are perpendicular, then $\phi=\pi/2$, $\sin\phi=1$, and the magnitude of their cross product is equal to the product of the magnitudes of the two vectors and the direction of the cross product is given by the right-hand rule. In particular $\mathbf{i}\times\mathbf{j}$ is in the direction of **k** (rotate **i** into **j** with the fingers of your right hand and watch your thumb) and has a magnitude of unity. But this is just the unit vector **k**. Thus, in a similar way,

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{i}=-\mathbf{k} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j} \qquad \mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i}$$

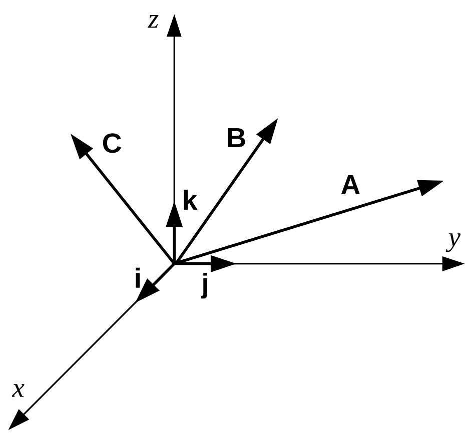

Figure C.3 $\mathbf{C}=\mathbf{A}\times\mathbf{B}$

Let us write the cross product

$$\begin{aligned}\mathbf{C} &= \mathbf{A}\times\mathbf{B}\\ &= \left(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}\right)\times\left(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}\right)\\ &= \left(A_xB_x\mathbf{i}\times\mathbf{i}\right)+\left(A_xB_y\mathbf{i}\times\mathbf{j}\right)+\left(A_xB_z\mathbf{i}\times\mathbf{k}\right)\\ &\quad+\left(A_yB_x\mathbf{j}\times\mathbf{i}\right)+\left(A_yB_y\mathbf{j}\times\mathbf{j}\right)+\left(A_yB_z\mathbf{j}\times\mathbf{k}\right)\\ &\quad+\left(A_zB_x\mathbf{k}\times\mathbf{i}\right)+\left(A_zB_y\mathbf{k}\times\mathbf{j}\right)+\left(A_zB_z\mathbf{k}\times\mathbf{k}\right)\end{aligned}$$

Using the above results to evaluate the cross products of the unit vectors, we can write

$$\begin{aligned}\mathbf{C} &= \mathbf{A}\times\mathbf{B}\\ \mathbf{C} &= \left(A_yB_z-A_zB_y\right)\mathbf{i}+\left(A_zB_x-A_xB_z\right)\mathbf{j}+\left(A_xB_y-A_yB_x\right)\mathbf{k}\\ &= C_x\mathbf{i}+C_y\mathbf{j}+C_z\mathbf{k}\end{aligned}$$

Therefore, if $\mathbf{C}=\mathbf{A}\times\mathbf{B}$, the components of **C** are given in terms of the components of **A** and **B** by

$$\begin{aligned}C_x &= A_yB_z-A_zB_y\\ C_y &= A_zB_x-A_xB_z\\ C_z &= A_xB_y-A_yB_x\end{aligned}$$

A useful way to remember the components of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is to recognize that **C** can be written as the determinant

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{C.3}$$

If you evaluate this determinant, you obtain the same result

$$\mathbf{C} = \left(A_y B_z - A_z B_y\right)\mathbf{i} + \left(A_z B_x - A_x B_z\right)\mathbf{j} + \left(A_x B_y - A_y B_x\right)\mathbf{k} \tag{C.4}$$

C.4 Associative Law

The associative law does not in general hold for the vector product. Thus

$$\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) \neq \left(\mathbf{A} \times \mathbf{B}\right) \times \mathbf{C}$$

as can be seen by the simple example shown in Fig. C.4. Since **A** and **B** are parallel, $\left(\mathbf{A} \times \mathbf{B}\right) \times \mathbf{C} = 0$. $\left(\mathbf{B} \times \mathbf{C}\right)$ is a vector directed along the +*z* axis (out of the paper), however, so that $\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right)$ is a nonzero vector directed along the -y axis.

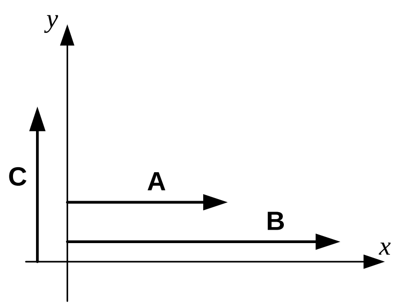

Figure C.4 $\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) \neq \left(\mathbf{A} \times \mathbf{B}\right) \times \mathbf{C}$

C.5 Cross Product Geometric Properties

Consider the parallelogram with its sides formed by the vectors **A** and **B** as shown in Fig. C.5. The area of the parallelogram is $h|\mathbf{A}|$ where $h = |\mathbf{B}| \sin\phi$. Recall that the magnitude of the cross product is given by

$$\left|\mathbf{A} \times \mathbf{B}\right| = |A||B| \sin\phi$$

Therefore in terms of **A** and **B** the area of the parallelogram is given by $|\mathbf{A}\times\mathbf{B}|$.

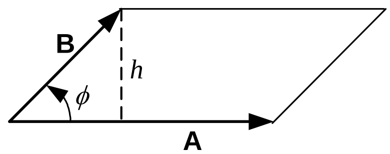

Figure C.5 The area of the parallelogram is $|\mathbf{A}\times\mathbf{B}|$

Consider the parallelepiped with its sides formed by the vectors **A**, **B**, and **C** as shown in Fig. C.6. The volume of the parallelepiped is (area of parallelogram formed by **A** and **B**) (height *h*) $=(|\mathbf{A}\times\mathbf{B}|)(\mathbf{C}\bullet\mathbf{n})$, where **n** is a unit vector parallel to $\mathbf{A}\times\mathbf{B}$. Since $\mathbf{A}\times\mathbf{B}=|\mathbf{A}\times\mathbf{B}|\mathbf{n}$, then in terms of **A**, **B**, and **C** the volume of the parallelepiped is given by $(\mathbf{A}\times\mathbf{B})\bullet\mathbf{C}$.

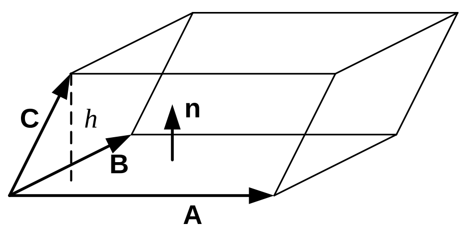

Figure C.6 The area of the parallelepiped is $(\mathbf{A}\times\mathbf{B})\bullet\mathbf{C}$

C.6 Scalar Triple Product

The expression $\mathbf{D}\bullet(\mathbf{A}\times\mathbf{B})$ is called the *scalar triple product*. It can be evaluated in terms of the rectangular components of **A**, **B**, and **D** by letting $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ as in Example 5c. Then

$$\mathbf{D}\bullet(\mathbf{A}\times\mathbf{B})=\mathbf{D}\bullet\mathbf{C}$$
$$\mathbf{D}\bullet(\mathbf{A}\times\mathbf{B})=D_x\left(A_yB_z-A_zB_y\right)+D_y\left(A_zB_x-A_xB_z\right)+D_z\left(A_xB_y-A_yB_x\right)$$

Note that the scalar triple product can be written as the determinant

$$\mathbf{D}\bullet\mathbf{A}\times\mathbf{B}=\begin{vmatrix} D_x & D_y & D_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{(C.5)}$$

C.7 Example C1

Given the vectors $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ find $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$.

Answer 1:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \mathbf{i}(4-4) + \mathbf{j}(12+2) + \mathbf{k}(1+6)$$

$$= 14\mathbf{j} + 7\mathbf{k}$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{196+49} = \sqrt{245} = 15.7$$

Answer 2: In Matlab the cross product of vectors **A** and **B** can be written as *cross*(**A,B**) as shown in Matlab Example C1.

Matlab Example C1

```
>> A = [1 -2 4]

A =
     1    -2     4

>> B = [3 1 -2]

B =
     3     1    -2

>> C = cross(A,B)

C =
     0    14     7

>> magC = norm(C)

magC =

   15.6525

>>
```

C.8 Example C2

Use $\left|\mathbf{A}\times\mathbf{B}\right| = AB\sin\phi$ to find the angle ϕ between **A** and **B** in Example 5g, and compare with the result of Example 4e.

Answer 1:

$$\sin\phi = \frac{\mathbf{A}\times\mathbf{B}}{AB}$$

$$= \frac{\sqrt{245}}{\sqrt{21}\sqrt{14}} = 0.915$$

$$\phi = 180^\circ - 66^\circ = 114^\circ$$

Answer 2: In Matlab the solution can be found by writing the single Matlab equation shown in Matlab Example C2.

Matlab Example C2

```
>> A = [1 -2 4]

A =

     1    -2     4

>> B = [3 1 -2]

B =

     3     1    -2

>> phi = 180 - (asin(norm(cross(A,B))/(norm(A)*norm(B))))*180/pi

phi =

  114.0948

>>
```

Note carefully the need to use parentheses in the equation for *phi*. The Matlab function *asin* for the arc sine gives the answer in radians. Thus, that result must be multiplied by 180/π to give the answer in degrees. Also note that the angle *phi* is greater than 90 degrees (as can be determined by plotting the vectors) and therefore the arc sine result must be subtracted from 180 degrees.

Problems

Where appropriate use Matlab to find the answers to the following problems.

C-1 If $\mathbf{A} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, find

(a) $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$

(b) $|\mathbf{A} \times \mathbf{B}|$

(c) $\sin\phi$ and ϕ where ϕ is the smaller angle between **A** and **B**.

C-2 If $\mathbf{A} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, find $\mathbf{A} \times \mathbf{B}$, $(\mathbf{A} \times \mathbf{B}) \bullet \mathbf{B}$, and $(\mathbf{A} \times \mathbf{B}) \bullet \mathbf{A}$.

C-3 If $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

(b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

C-4 Evaluate

(a) $2\mathbf{i} \times (3\mathbf{j} - 4\mathbf{k})$

(b) $(\mathbf{i} + 2\mathbf{j}) \times \mathbf{k}$

(c) $(2\mathbf{i} - 4\mathbf{j}) \times (\mathbf{i} + \mathbf{k})$